Buckling and Vibration of Arches and Tied Arches

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Abstract: A simple method of computing the planar elastic buckling loads, natural frequencies, and corresponding mode shapes for arches and tied arches has been developed. The proposed method is applicable to arches and tied arches of general shape. Stiffnesses of arch rib and tie (if present) may vary in any manner along the span. The only major limitation in the applicability of the procedure is that the simplifying idealizations that are made can result in significant error for very steep arches. The details of the proposed method of analysis have been developed to be compatible with normal design office practice. The procedure involves linear elastic analysis with multiple loadings to obtain a simplified flexibility matrix, manual development of a stability matrix (for buckling) or mass matrix (for vibration), and solution of an eigenvalue equation. Several examples that are presented indicate that the proposed method is accurate—at least to the degree normally required in structural design—for arches and tied arches that have rise/span ratios within the range customarily encountered in medium- and long-span bridges.

Introduction

Most structural design offices today have the means to perform accurate linear elastic structural analysis using a digital computer. This paper presents a simple method of using this linear analysis capability—without additional computer programming—to determine the critical loads and modes for planar elastic buckling of arches and tied arches. The arch does not have to be of any regular shape except that the rib must not be vertical or near-vertical at any location. Stiffnesses can vary in any manner.

These features of the proposed method of analysis make it more readily applicable to most real structures and more convenient in many typical design-office situations than other techniques described in the literature (1,3,4).

The proposed technique is applicable to arches and tied arches. Tied arches have received little attention in the literature on stability, possibly because of the widespread belief that tied arches are not susceptible to planar elastic buckling. This belief is reflected in current bridge design specifications (2). A simple proof that tied arches can be unstable is presented in Appendix I.

The procedure suggested for determination of critical loads and modes for buckling can be extended very easily to include computation of natural frequencies and modes for planar vibration of arches and tied arches.

Definition of Structure

A tied arch is shown in Fig. 1. There is no restriction on the shape of the arch rib or on the distribution of stiffness along the rib and tie. Ver-

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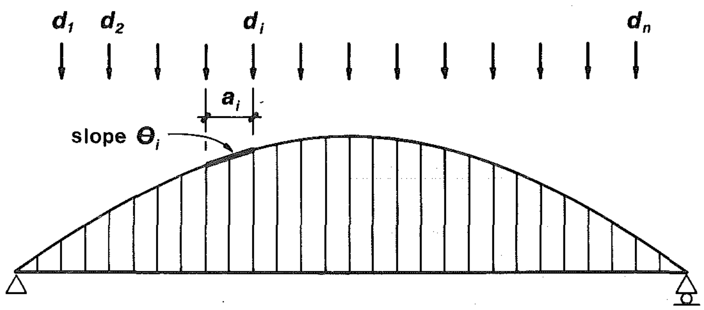

FIG. 1.—Definition of Structure

tical displacement components d_1, d_2, ..., d_n are defined at n locations along the span. The horizontal distance between components d_{i-1} and d_i is defined as a_i. The slope of the arch rib in this distance is defined as θ_i. The displacement components should be sufficiently close together that the arch rib can be idealized as being made up of straight segments between the locations of these components. The distances a_1, a_2, ..., a_{n+1} do not have to be equal. There does not have to be a d component at each hanger location.

For an arch without a tie, the vertical displacement components, horizontal distances, and slopes are defined in exactly the same way as for tied arches.

A "bifurcation" buckling solution requires that there be no displacements at load intensities less than the critical value. To achieve this idealized condition in an arch or tied arch, length changes in rib, tie, and hangers must be neglected and the load configuration must be such that it causes no deformation (at intensities less than the critical value), i.e., the loading must be "funicular." The horizontal component of the compression in the arch rib caused by this loading is defined as H. The tension in the tie of a tied arch would also be H. The critical values of H at which the structure becomes unstable, together with the corresponding buckling modes, will be determined in this analysis.

Limitation.—The effect of horizontal displacement of points on the arch rib is not modeled accurately in the proposed analysis. The resulting errors could be significant for very steep arches. The examples studied indicate that the proposed technique is sufficiently accurate for typical engineering design purposes if the maximum slope of the arch rib is no steeper than 45°.

Second-Order Forces

Consider segment i of the rib and tie (Fig. 1). If displacements d_{i-1} and d_i occur at the two ends of segment i, the corresponding changes in slopes and forces are as follows:

1. Rib slope changes from θ_i to $\theta_i + (d_{i-1} - d_i)/a_i$.
2. Compression in rib changes from $H \sec \theta_i$ to $H \sec [\theta_i + (d_{i-1} - d_i)/a_i]$.

3. Vertical component of compression in rib changes from $H \tan \theta_i$ to $H \tan [\theta_i + (d_{i-1} - d_i)/a_i]$.

4. The slope changes from 0 to $(d_{i-1} - d_i)/a_i$.

5. Tension in tie remains H.

6. Vertical component of tension in tie changes from 0 to $H(d_{i-1} - d_i)/a_i$.

7. Net vertical component in panel i, upward positive at node i, changes from $H \tan \theta_i$ to $H\{\tan [\theta_i + (d_{i-1} - d_i)/a_i] - (d_{i-1} - d_i)/a_i\}$.

The increase in the net vertical component in panel i, upward positive at node i, is $H\{\tan [\theta_i + (d_{i-1} - d_i)/a_i] - [(d_{i-1} - d_i)/a_i] - \tan \theta_i\}$. By trigonometric manipulation, this expression can be simplified to $H \tan^2 \theta_i (d_{i-1} - d_i)/a_i$.

Similarly, the increase in the net vertical component in panel $i + 1$, downward positive at node i, is $H \tan^2 \theta_{i+1}(d_i - d_{i+1})/a_{i+1}$.

The additional downward force at node i, $\bar{p}_i$, required to maintain equilibrium by balancing the effect of the displacements is given by:

$$\bar{p}_i = H\left[\tan^2 \theta_i \frac{(d_{i-1} - d_i)}{a_i} - \tan^2 \theta_{i+1} \frac{(d_i - d_{i+1})}{a_{i+1}}\right] \qquad (1)$$

For an arch without a tie, the equation for $\bar{p}_i$ is identical to Eq. 1 except that $\tan^2 \theta_i$ and $\tan^2 \theta_{i+1}$ are replaced by $\sec^2 \theta_i$ and $\sec^2 \theta_{i+1}$, respectively. The difference between the tied arch and the arch without a tie will become clearer if it is noted that $\tan^2 \theta = \sec^2 \theta - 1$. Thus, the effect of the tie is to reduce terms in the equation for second-order forces from $\sec^2 \theta$ to $\sec^2 \theta - 1$.

For both the tied arch and the arch without a tie, the equation for second-order forces can be rewritten in matrix notation as follows:

$$\{\bar{p}\} = H[G]\{d\} \qquad (2)$$

in which, for a tied arch:

$$G_{i,i-1} = \frac{\tan^2 \theta_i}{a_i} \qquad (3a)$$

$$G_{i,i} = \frac{-\tan^2 \theta_i}{a_i} - \frac{\tan^2 \theta_{i+1}}{a_{i+1}} \qquad (3b)$$

$$G_{i,i+1} = \frac{\tan^2 \theta_{i+1}}{a_{i+1}} \qquad (3c)$$

and, for the arch without a tie:

$$G_{i,i-1} = \frac{\sec^2 \theta_i}{a_i} \qquad (4a)$$

$$G_{i,i} = \frac{-\sec^2 \theta_i}{a_i} - \frac{\sec^2 \theta_{i+1}}{a_{i+1}} \qquad (4b)$$

$$G_{i,i+1} = \frac{\sec^2 \theta_{i+1}}{a_{i+1}} \qquad (4c)$$

and all other elements of $[G]$ are zero.

Linear Force-Displacement Relationship

The externally applied force corresponding to displacement d_i is denoted at p_i. The linear flexibility matrix that relates $\{d\}$ and $\{p\}$, neglecting second-order effects, is defined by the following equation:

$$\{d\} = [F]\{p\} \qquad (5)$$

The $n \times n$ flexibility matrix, $[F]$, can be determined by performing a linear elastic analysis of the structure for n separate loading conditions. The n loadings should consist of unit values of p_i with $i = 1, 2, \ldots, n$. Element $F_{j,i}$ in the flexibility matrix is the displacement d_j caused by unit load p_i.

This method of determining the flexibility matrix is very convenient in the design office. It does not require any special computer programming. Any standard frame analysis program can be used for the linear elastic analysis. The joints and members in the linear analysis should be defined in such a way as to model the structure accurately; the joints do not have to be limited to the locations at which the displacements d_i ($i = 1, 2, \ldots, n$) have been defined.

Flexural and axial stiffnesses of members should be modeled accurately in the linear analysis. This is inconsistent with the assumptions in the buckling analysis, which neglects axial deformations in order to obtain an idealized bifurcation solution. The effect of this inconsistency is small, as will be shown in the examples of use of the proposed technique.

Buckling Solution

The overall force-displacement relationship for the structure, including second-order effects, is as follows:

$$\{d\} = [F](\{p\} + \{\bar{p}\}) \qquad (6)$$

Substituting Eq. 2 in Eq. 6:

$$\{d\} = [F]\{p\} + H[F][G]\{d\} \qquad (7)$$

To obtain the idealized bifurcation buckling solution, prebuckling or first-order displacements must be taken as zero. Thus, $[F]\{p\}$ is taken as zero and the following is obtained:

$$\{d\} = H[F][G]\{d\} \qquad (8)$$

Eq. 8 is an eigenvalue problem, which can be solved to obtain the critical values of the horizontal thrust, H, and the corresponding buckling modes. Simple computer programs for eigenvalue solution are readily available.

Vibration

To determine the natural frequencies and modes for free vibration of the arch or tied arch, the mass of the structure must be idealized as being concentrated or "lumped" at the locations of the n displacement components. A mass matrix, $[M]$, is defined as an $n \times n$ matrix in which all nondiagonal elements are zero and the ith diagonal element is the mass at the location of displacement component d_i.

From the fundamental principals of dynamic analysis, it can be shown that the equation for free vibration of the structure is as follows:

$$\{d\} = \omega^2[F][M]\{d\} \qquad (9)$$

Eq. 9 is an eigenvalue problem, which can be solved to obtain the natural frequencies, ω, and the corresponding vibration modes.

Examples of Application

Several examples have been analyzed to demonstrate the application of the proposed techniques and to evaluate their accuracy and ease of use. The structure in Examples 4 and 5 is based on the tied arch in a recently completed highway bridge. The proposed methods of buckling and vibration analysis were used by the writer in the design of this bridge, which has a tied arch span of 909 ft (277 m).

Example 1

The arch in this example (see Fig. 2) does not have a tie. It is a parabolic steel arch of constant cross section, without hinges. Exact buckling solutions for arches of this type are available in the literature. The span is 910 ft (278 m) and the rise is 182 ft (55.6 m). The cross-sectional area is 6.61 ft^2 (0.614 m^2) and the moment of inertia is 129 ft^4 (1.12 m^4). The weight for vibration analysis was taken as 11 kips per foot of horizontal length (160 kN per meter of horizontal length).

For linear analysis the arch was idealized as 20 straight members. The joints between the members are denoted by filled circles in Fig. 2. For the buckling and vibration analyses, 9 displacement components were established, as indicated by the arrows numbered 1 through 9 in Fig. 2. Since the structure is symmetric, determination of the 9 × 9 flexibility matrix required only 5 loading conditions in the linear analysis.

Example 2

The structure in Example 2 is the same as in Example 1. The idealization for linear analysis was also the same as in Example 1. For the

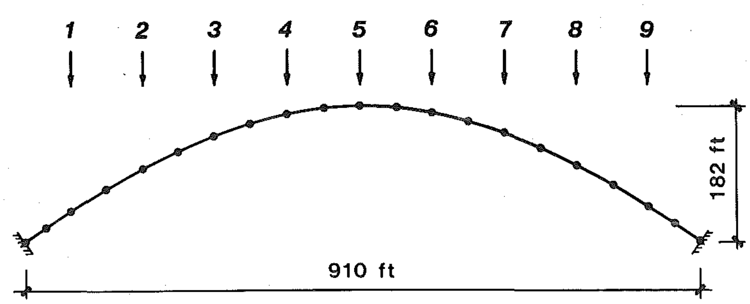

FIG. 2.—Structure Analyzed in Example 1 (1 ft = 0.305 m)

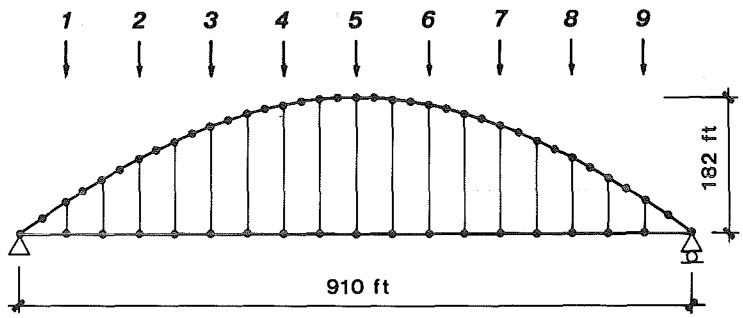

FIG. 3.—Structure Analyzed in Example 4 (1 ft = 0.305 m)

buckling and vibration analyses, the number of displacement components was increased from 9 to 17.

Example 3

The structure in this example is the same as in Examples 1 and 2 except that the cross-sectional area of the arch rib is increased four-fold. The analysis details were the same as in Example 2.

Example 4

This is a steel tied arch with a parabolic rib (see Fig. 3). The span is 910 ft (278 m) and the rise is 182 ft (55.6 m). There are 17 hangers between the rib and tie. The hangers are 49 ft (14.9 m) apart; the outermost hangers are 63 ft (19.2 m) from the ends of the span.

The rib and tie cross sections are not constant. The rib is largest at the ends of the span, where its area is 2.94 ft^2 (0.273 m^2) and its moment of inertia is 15.8 ft^4 (0.136 m^4). The tie is largest near the quarter points of the span, where its area is 3.67 ft^2 (0.341 m^2) and its moment of inertia is 113 ft^4 (0.98 m^4). The minimum area and moment of inertia of the rib and tie are about 3/4 of the maximum values. The hangers are multiple steel cables; the equivalent solid steel area of each hanger is 0.093 ft^2 (0.0077 m^2). For vibration analysis, the total weight of the structure plus material supported on it varied from about 11 kips per foot of horizontal length (160 kN per meter of horizontal length) near the ends of the span to about 90% of this amount near midspan.

For linear analysis, the rib was idealized as 36 straight members. The joints in the analysis are denoted by filled circles in Fig. 3. (The joints are not hinges; the rib and tie are continuous through the joints.) For the buckling and vibration analyses, 9 displacement components were established, as indicated by the arrows numbered 1 through 9 in Fig. 3. Determination of the 9×9 flexibility matrix required only 5 loading conditions in the linear analysis (because of the symmetry of the structure).

Example 5

The structure in Example 5 is the same as in Example 4. The idealization for linear analysis was also the same as in Example 4. For the buckling and vibration analyses, the number of displacements was increased from 9 to 17. (A displacement component was established at each hanger location.)

Example 6

The structure in this example is the same as in Examples 4 and 5 except that the cross-sectional area of rib, tie, and hangers is increased four-fold. The analysis details were the same as in Example 5.

Results of Example Analyses

The critical horizontal thrust, H, computed for the first two modes of buckling in each of the six examples is presented in Table 1. The natural periods for the first two modes of vibration are also tabulated. (The tabulated periods, in seconds per cycle, are equal to $2\pi/\omega$.) The buckling and vibration mode shapes for Examples 1 and 4 are shown in Figs. 4 and 5, respectively. In all six examples, the first and third modes for both buckling and vibration were found to be antisymmetric while the second mode is symmetric.

Effect of Number of Displacement Components.—Examples 2 and 5 are identical with Examples 1 and 4, respectively, except that 1 and 4 were analyzed using 9 displacement components while 17 components were used in Examples 2 and 5. It is clear from the results presented in Table 1 that the increase in the number of displacement components, n, had only a modest effect on the results of the stability analysis and essentially no effect on the results of the vibration analysis. A further increase in n, beyond 17, could be expected to have very little effect on the stability results. The solutions with $n = 9$ should be adequate for most purposes.

Effect of Axial Stiffness.—There is an inconsistency in the proposed method of stability analysis in that the buckling solution neglects axial deformations (in order to obtain an idealized "bifurcation" solution), while the linear analysis to obtain the flexibility matrix does include axial deformations. Examples 3 and 6 (which are identical with 2 and 5 except for a four-fold increase in the axial stiffness of all members) are intended to evaluate the importance of this inconsistency.

TABLE 1.—Critical Loads and Natural Periods in Example Analyses

Example (1)	Type of arch (2)	Number of displacement components, n (3)	Axial stiffness of members (4)	Critical Horizontal Thrust, H (kips × 10^3) Mode 1 (5)	Critical Horizontal Thrust, H (kips × 10^3) Mode 2 (6)	Natural Period of Vibration (sec) Mode 1 (7)	Natural Period of Vibration (sec) Mode 2 (8)
1	Fixed	9	Normal	44.7	78.9	2.24	1.25
2	Fixed	17	Normal	43.0	73.2	2.24	1.26
3	Fixed	17	High	43.2	73.3	2.24	1.25
4	Tied	9	Normal	87.0	200.9	3.36	1.71
5	Tied	17	Normal	83.4	188.9	3.39	1.72
6	Tied	17	High	85.8	204.5	3.35	1.65

Note: 1 kip = 4.45 kN.

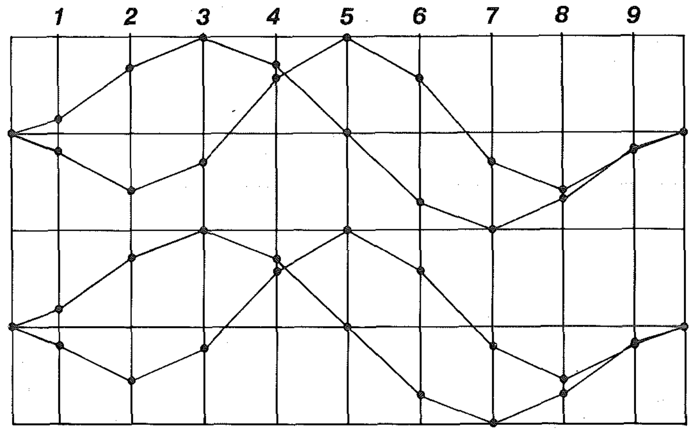

FIG. 4.—First Two Buckling Modes (Upper Curves) and Vibration Modes (Lower Curves) in Example 1

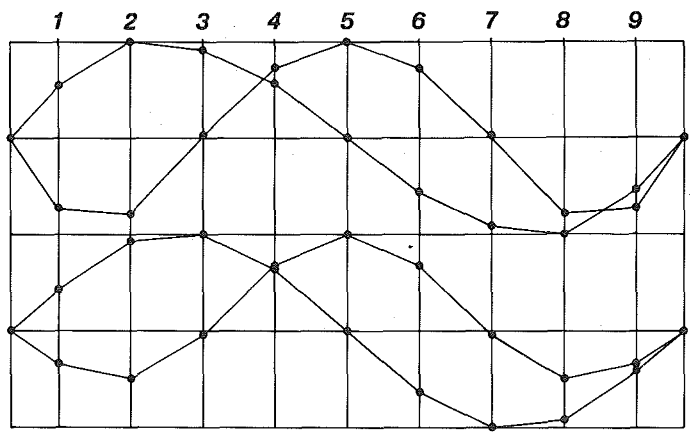

FIG. 5.—First Two Buckling Modes (Upper Curves) and Vibration Modes (Lower Curves) in Example 4

In the fixed arch, rib axial stiffness has virtually no effect on the buckling solution (see Table 1). In the tied arch, the axial stiffnesses of rib, tie, and hangers appear to have a slight effect on the first-mode buckling solution and a moderate effect on the second-mode solution.

Comparison with Exact Solution.—Exact buckling solutions are available in the literature (3) for parabolic arches of constant stiffness. For the arch in Examples 1, 2, and 3, the exact solution for critical horizontal thrust, H, is 41.1×10^3 kips for the first mode of buckling. Thus, the error in Examples 2 and 3 is about 5%. This error would not be reduced significantly by using a larger number of displacement components in the analysis. The 5% error occurs because the effect of horizontal displacement of points on the arch rib is not modeled accurately in the proposed method of analysis. The magnitude of the error would be greater for steeper arches and smaller for shallower arches.

Summary of Suggested Procedure

The suggested procedure for determining the buckling loads, vibration frequencies, and corresponding modes for arches and tied arches consists of the following steps:

1. Idealize the structure for linear analysis (to be performed using any general frame analysis computer program). The computational model selected here could be the same as that used for detailed design of the structure.
2. Select n and establish n vertical displacement components for buckling and vibration analysis. Between 8 and 15 components should be adequate in most cases.
3. Determine the $n \times n$ flexibility matrix, $[F]$, by performing a linear analysis of the structure for n separate loading conditions, each of which is a unit load along one of the selected displacement coordinates. (For symmetric structures, the number of loadings can be reduced by only applying those on half of the structure.)
4. For buckling analysis, determine nonzero elements in the stability matrix, $[G]$, using Eq. 3 or 4. For vibration analysis, determine the mass matrix, $[M]$, by computing the mass tributary to each of the n displacement components.
5. Solve the nth-order eigenvalue problem, Eq. 8 for buckling or Eq. 9 for vibration, to obtain the critical values of horizontal thrust, H, or the natural frequencies, ω, and the corresponding mode shapes.

Conclusions

The proposed technique for buckling and vibration analysis is applicable to most arches and tied arches. The details of the procedure are compatible with normal design office practice and do not require special programming or computational expertise. The proposed methods are accurate—at least to the degree normally required in structural design—for arches and tied arches that have rise/span ratios within the range customarily utilized in medium- and long-span bridges.

The general analytical concept used in this work consists of linear analysis with multiple loadings to obtain a reduced flexibility matrix, followed by use of this flexibility matrix with a stability or mass matrix in an eigenvalue solution. This concept is applicable to a wide range of stability and vibration problems.

Appendix I.—Simple Proof that Tied Arches Can Be Unstable

A highly idealized four-panel tied arch is shown in Fig. 6(a). There are hangers (and loading points) at midspan and at quarter points. The rib is straight between hanger locations. To simplify the structure further, the load at the quarter points is defined as being zero and, consequently, there is no change in the slope of the rib at these points in

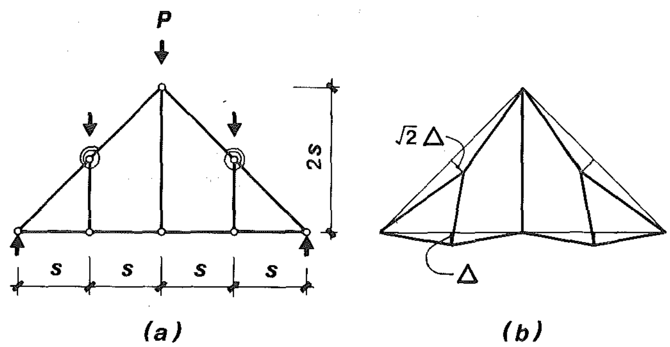

FIG. 6.—Idealized Tied Arch for Proof of Susceptibility to Buckling: (*a*) Undeformed Configuration; and (*b*) Displacements

the undeformed structure. The span is $4s$; the rise is $2s$; the downward load at midspan is P. All members (rib, tie, hangers) are considered to be hinged at all intersection points, as denoted by circles in Fig. 6(*a*). The hangers and the rib and tie segments between hinges are perfectly rigid. There are rotational springs of stiffness k (moment per radian) between the rib segments at the quarter-point hinges.

This idealized structure bears little physical resemblance to most real tied arches. However, it should serve as a convenient model for examining the validity of the idea that tied arches are not susceptible to overall planar buckling. The arguments for the tied arch's immunity from buckling are applicable also to this idealized structure.

The literature on buckling of arches (1) suggests that the structure in Fig. 6(*a*) should not be susceptible to buckling, since any disturbing effect due to deformation of the rib would be counteracted by an equal "restoring" effect at the tie. As demonstrated by the following analysis, this reasoning is not strictly valid, in that the restoring effect of the tie reduces, but does not totally eliminate, the arch's susceptibility to buckling.

Consider a set of displacements imposed on the structure as shown in Fig. 6(*b*). The rib quarter points are displaced by $\sqrt{2}\Delta$ from the straight lines connecting the apex and the ends of the rib. The resulting downward displacement of the quarter points in the tie is shown as Δ in Fig. 6(*b*). (The tie displacement would actually be slightly less than Δ owing to the tilt of the hanger. Ignoring this effect, which is of a smaller order of magnitude, results in an upper-bound value of tie displacement, an upper-bound accounting of the "restoring" effect of the tie, and an upper bound of buckling load.)

As a result of displacements Δ at the quarter points, the overall chord length of the tie changes from $4s$ to $4s \cos(\Delta/s)$.

As a result of displacement $\sqrt{2}\Delta$ at the quarter points of the rib, the chord length of the rib between the apex and the point of connection to the tie changes from $2\sqrt{2}s$ to $2\sqrt{2}s \cos(\Delta/s)$.

Thus, all three sides of the triangle formed by the apex and the two support points are reduced in length by a factor of $\cos(\Delta/s)$. It follows that the height of the apex above the supports changes from $2s$ to $2s \cos(\Delta/s)$. Thus, the point of application of load P moves downward by an amount equal to $2s[1 - \cos(\Delta/s)]$. For small values of (Δ/s), $\cos(\Delta/s)$ approaches $[1 - (\Delta/s)^2/2]$ and $2s[1 - \cos(\Delta/s)]$ approaches Δ^2/s.

If the displacements shown in Fig. 6(b) represent buckling under load P, the work done by load P must be equal to the work done on the springs. The critical value of the load, P_c, at which buckling occurs can be determined as follows:

$$\text{Distance moved by load} = \frac{\Delta^2}{s} \quad (10a)$$

$$\text{Work done by load} = \frac{P\Delta^2}{s} \quad (10b)$$

$$\text{Rotation of spring} = \frac{2\Delta}{s} \quad (10c)$$

$$\text{Work done by two springs} = \frac{2k\left(\frac{2\Delta}{s}\right)^2}{2} = \frac{4k\Delta^2}{s^2} \quad (10d)$$

$$\text{For buckling:} \quad \frac{P_c\Delta^2}{s} = \frac{4k\Delta^2}{s^2} \quad (10e)$$

$$P_c = \frac{4k}{s} \quad (10f)$$

It can be shown that if the tie and hangers are eliminated from the structure in Figs. 6(a and b) and the ends of the rib are pinned to nonmovable supports, the critical load changes to $2k/s$. The effect of the tie, therefore, is to double the structure's resistance to buckling in this instance. In general, the proportionate increase in buckling resistance caused by a tie will be greater for shallower arches and smaller for steeper arches. In all cases, however, the tied arch is susceptible to buckling, though at a load greater than the critical load for an arch of equal geometry and stiffness without a tie.

Appendix II.—References

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Appendix III.—Notation

The following symbols are used in this paper:

a_i = horizontal distance between displacement coordinates $i - 1$ and i;
d_i = vertical displacement along ith coordinate;
$[F]$ = flexibility matrix relating $\{p\}$ and $\{d\}$;
$[G]$ = stability matrix (defined in Eq. 2);
H = horizontal component of compression in arch rib;
$[M]$ = mass matrix;
n = number of vertical displacement coordinates;
p_i = externally applied force along ith coordinate;
$\bar{p}_i$ = second-order force along ith coordinate;
θ_i = slope of arch rib between location of coordinates $i - 1$ and i; and
ω = vibration frequency in radians per unit time.